University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 1

1. Two supernovæ are observed on earth in the direction of the north star, separated by 10 years. From the theory of supernovæ these are known to have identical ("standard candle") light output, yet the first is observed to have four times the light intensity of the second because it is closer.
(a.)
An astronomer theorizes that the two stars were at rest with respect to the earth, and that the first supernova triggered the second. What is the *maximum* distance between the earth and the first supernova under this hypothesis?
(b.)
A physicist theorizes that the two stars were traveling with the same (unspecified) velocity away from the earth, and that, in their common rest frame, the two supernovæ occurred at the same proper time. What is the *minimum* distance between the earth and the first supernova under this hypothesis?

2. Inertial reference frames $\mathcal{S}'$ and $\mathcal{S}$ coincide at $t' = t = 0$. You may ignore the z dimension, so that a point in spacetime is determined by only three quantities $r \equiv (ct, x, y)$. The Lorentz transformation between $\mathcal{S}$ and $\mathcal{S}'$ is given by

$$\begin{pmatrix} ct' \\ x' \\ y' \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct \\ x \\ y \end{pmatrix},$$

where $\mathcal{L}$ is a 3×3 matrix.
(a.)
Assume for this part that $\mathcal{S}'$ moves with velocity

$$\mathbf{V} = \beta c \hat{\mathbf{x}}$$

with respect to $\mathcal{S}$. Using your knowledge of Lorentz transformations (no derivation necessary), write $\mathcal{L}$ for this case.
(b.)
Assume for this part that $\mathcal{S}'$ moves with velocity

$$\mathbf{V} = \beta c \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}$$

with respect to $\mathcal{S}$. Find $\mathcal{L}$ for this case. (*Hint.* Rotate to a system in which $\mathbf{V}$ is along the $\hat{\mathbf{x}}$ axis, transform using your answer for part (**a.**), and then rotate back. Check that your result is symmetric under interchange of x and y, as is $\mathbf{V}$, and that it reduces to the unit matrix as $\beta \to 0$.)

3. Work out the Lorentz transformation matrix $\mathcal{L}$ for the general case in which $\boldsymbol{\beta}$ of frame $\mathcal{S}'$ is directed along an arbitrary unit vector $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ as seen in frame $\mathcal{S}$, *e.g.*

$$r' = \mathcal{L}r, \quad \mathcal{L} = ?$$

4. In a straight channel oriented along the $\hat{\mathbf{z}}$ axis there are two opposing beams:

- a beam of positrons (charge $+e$) with velocity $+\hat{\mathbf{z}}\beta c$.
- a beam of electrons (charge $-e$) with velocity $-\hat{\mathbf{z}}\beta c$.

Each beam is confined to a small cylindrical volume of cross sectional area A centered on the $\hat{\mathbf{z}}$ axis. Within that volume, there is a uniform number density $= n$ positrons/m^3 and n electrons/m^3.
(a.)
In terms of n, A, e, and β, calculate the total current I in the channel due to the sum of both beams (note $I \neq 0$).
(b.)
Use Ampère's Law to calculate the azimuthal magnetic field B_ϕ outside the channel a distance r from the $\hat{\mathbf{z}}$ axis.

Consider now a Lorentz frame $\mathcal{S}'$ travelling in the $\hat{\mathbf{z}}$ direction with velocity βc relative to the lab frame described above. (This β is the same β as above.)
(c.)
As seen in $\mathcal{S}'$, calculate the number density n'_+ of *positrons* within the cylindrical volume. (You may use elementary arguments involving

space contraction, or you may use the fact that $(c\rho, \mathbf{j})$ is a 4-vector, where ρ is the charge density (Coul/m^3) and $\mathbf{j}$ is the current density (amps/m^2).)

(d.)

As seen in $\mathcal{S}'$, calculate the number density n'_- of *electrons* within the cylindrical volume.

(e.)

Calculate the radial electric field E'_r seen in $\mathcal{S}'$. Do this both

- by using the results of (c.) and (d.) plus Gauss's law, and
- by using the results of (b.) plus the rules for relativistic $\mathbf{E}$ and $\mathbf{B}$ field transformations.

5. (Taylor and Wheeler problem 51)

The clock paradox, version 3.

Can one go to a point 7000 light years away – *and return* – without aging more than 40 years? "Yes" is the conclusion reached by an engineer on the staff of a large aviation firm in a recent report. In his analysis the traveler experiences a constant "1-g" acceleration (or deceleration, depending on the stage reached in her journey). Assuming this limitation, is the engineer right in his conclusion? (For simplicity, limit attention to the first phase of the motion, during which the astronaut accelerates for 10 years – then double the distance covered in that time to find how far it is to the most remote point reached in the course of the journey.)

(a.)

The acceleration is *not* $g = 9.8$ meters per second per second relative to the laboratory frame. If it were, how many times faster than light would the spaceship be moving at the end of ten years (1 year $= 31.6 \times 10^6$ seconds)? *If the acceleration is not specified with respect to the laboratory, then with respect to what is it specified?* Discussion: Look at the bathroom scales on which one is standing! The rocket jet is always turned up to the point where these scales read one's *correct* weight. Under these conditions one is being accelerated at 9.8 meters per second per second with respect to a spaceship that (1) instantaneously happens to be riding alongside with identical velocity, but (2) is *not* being accelerated, and, therefore (3) *provides the* (momentary) *inertial frame of reference relative to which the acceleration is g.*

(b.)

How much velocity does the spaceship have after a given time? This is the moment to object to the question and to rephrase it. *Velocity βc* is not the simple quantity to analyze. The simple quantity is the *boost parameter η*. This parameter is simple because it is *additive* in this sense: Let the boost parameter of the spaceship with respect to the imaginary instantaneously comoving inertial frame change from 0 to $d\eta$ in an astronaut time $d\tau$. Then the boost parameter of the spaceship with respect to the *laboratory* frame changes in the same astronaut time from its initial value η to the subsequent value $\eta + d\eta$. Now relate $d\eta$ to the acceleration g in the instantaneously comoving inertial frame. In this frame $g\,d\tau = c\,d\beta = c\,d(\tanh\eta) = c\,\tanh(d\eta) \approx c\,d\eta$ so that

$$c\,d\eta = g\,d\tau$$

Each lapse of time $d\tau$ on the astronaut's watch is accompanied by an additional increase $d\eta = \frac{g}{c}d\tau$ in the boost parameter of the spaceship. In the laboratory frame the total boost parameter of the spaceship is simply the sum of these additional increases in the boost parameter. Assume that the spaceship starts from rest. Then its boost parameter will increase linearly with *astronaut* time according to the equation

$$c\eta = g\tau$$

This expression gives the boost parameter η of the spaceship in the *laboratory* frame at any time τ in the *astronaut's* frame.

(c.)

What laboratory distance x does the spaceship cover in a given astronaut time τ? At any instant the velocity of the spaceship in the laboratory frame is related to its boost parameter by the equation $dx/dt = c\tanh\eta$ so that the distance dx covered in *laboratory* time dt is

$$dx = c\,\tanh\eta\,dt$$

Remember that the time between ticks of the astronaut's watch $d\tau$ appear to have the larger value dt in the laboratory frame (time dilation) given by the expression

$$dt = \cosh\eta \, d\tau$$

Hence the laboratory distance dx covered in *astronaut* time $d\tau$ is

$$dx = c \tanh\eta \cosh\eta \, d\tau = c \sinh\eta \, d\tau$$

Use the expression $c\eta = g\tau$ from part b to obtain

$$dx = c \sinh\left(\frac{g\tau}{c}\right) d\tau$$

Sum (integrate) all these small displacements dx from zero astronaut time to a final astronaut time to find

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right) - 1\right]$$

This expression gives the laboratory *distance* x covered by the spaceship at any time τ in the astronaut's frame.
(d.)
Plugging in the appropriate numerical values, determine whether the engineer is correct in his conclusion reported at the beginning of this exercise.

6. Electrons ($mc^2 = 0.5 \times 10^6$ eV) are accelerated over a distance of 3.2 km from rest to a total energy of 5×10^{10} eV at SLAC (Stanford).
(a.)
To what boost η are the electrons ultimately brought?
(b.)
Assuming that the electrons are subjected to a uniform acceleration as observed in their comoving inertial frame, how many g's of acceleration do they feel?
(c.)
As observed in the lab, for what time interval is each electron in flight? What is the corresponding *proper* time interval? Evaluate the ratio of the two intervals (a sort of average γ factor).

7. (Taylor and Wheeler problem 75)
Doppler equations.
A photon moves in the xy laboratory plane in a direction that makes an angle ϕ with the x axis, so that its components of momentum are $p_x = p\cos\phi$, $p_y = p\sin\phi$, and $p_z = 0$.
(a.)
Use the Lorentz transformation equations for the momentum-energy 4-vector and the relation $E^2/c^2 - p^2 = 0$ for a photon to show that, in the rocket frame $\mathcal{S}'$ (moving with velocity $\beta_r c$ along the x, x' direction, and coinciding with the laboratory frame at $t = t' = 0$), the photon has an energy E' given by the equation

$$E' = E \cosh\eta_r (1 - \beta_r \cos\phi)$$

and moves in a direction that makes an angle ϕ' with the x' axis given by the equation

$$\cos\phi' = \frac{\cos\phi - \beta_r}{1 - \beta_r \cos\phi}$$

(b.)
Derive the inverse equations for E and $\cos\phi$ as functions of E', $\cos\phi'$, and β_r.
(c.)
If the frequency of light in the laboratory is ν, what is the frequency ν' of light in the rocket frame? This difference in frequency due to relative motion is called the *relativistic Doppler shift*. Do these equations enable one to tell in what frame the *source* of the photons is at rest?

8. Consider the following situation. A star is known, by means of external data, to be located instantaneously a distance D from an observer on earth. The external data do not tell us the rate of change of D with time.

In her measurements, the observer corrects for aberration caused by the local velocity of the earth's surface, due both to its daily rotation and its yearly orbit. Therefore we do not need to take into account these boring local phenomena in what follows.

After making these corrections, the observer sees that the star is undergoing angular motion $d\psi/dt$ across the sky, such that $D\,d\psi/dt = c$, where c is the speed of light.

Finally, the observer measures the wavelength spectrum of light from this star, and finds its features not to be redshifted or blueshifted at all – they are exactly where they would be if the star were perfectly at rest with respect to the observer.

Is it possible that this situation is physically reasonable? If so, what might be the true motion of the star with respect to the observer? If not, why not?